# Null Wilson loop with Lagrangian insertion in $\mathcal{N} = 4$ super-Yang-Mills theory

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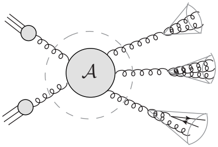
based on works 2202.05596

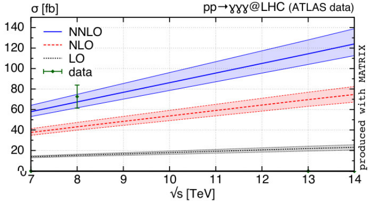
and 2204.00329 with Johannes Henn

18 May 2022

# The era of precision collider physics

- Ever improving experimental precision @ LHC
- Next-to-next-to-leading order theoretical predictions for QCD processes are required nowadays
- Great interest in $2 \to 3$ processes




plot from arXiv:2010.04681

- Finite parts of parton QCD scattering amplitudes are input for cross-section (and event-shape) calculations

$$\mathcal{A}_n^{\mathrm{QCD}} = \mathcal{Z}_{\mathrm{IR}} \times \mathcal{H}_n$$

# (Planar) maximally supersymmetric Yang-Mills theory is a theoretical laboratory for amplitude calculations

Established for $\mathcal{N} = 4$ sYM amplitudes and applied to QCD amplitudes

- on-shell recursion relations
  [Britto, Cachazo, Feng, Witten '05]
- generalized unitarity
  [Bern, Dixon, Dunbar, Kosower '94]
- symbols and alphabets
  [Goncharov, Spradlin, Vergu, Volovich '10]
- differential equations [Henn '13]
- pure integrals and leading singularities
  [Arkani-Hamed, Cachazo, Cheung, Kaplan '09]
  [Arkani-Hamed, Bourjaily, Cachazo, Trnka '10]
- color-kinematics duality
  [Bern, Carrasco, Johansson '08]

# QCD amplitudes vs planar $\mathcal{N} = 4$ sYM amplitudes

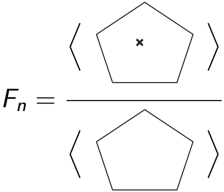
| finite part of $\mathcal{A}_n$ | QCD | planar $\mathcal{N} = 4$ sYM |
|---|---|---|
| # variables | $3n - 11$ | $3n - 15$ |

Hidden symmetries of the planar $\mathcal{N} = 4$ sYM severely restrict the form of $n$-particle amplitude

| $n$ | QCD | planar $\mathcal{N} = 4$ sYM |
|---|---|---|
| 4 | 1 | 0 |
| 5 | 4 | 0 |
| 6 | 7 | 3 |
| 7 | 10 | 6 |

Could we define an observable in planar $\mathcal{N} = 4$ sYM which resembles more closely (finite parts of) QCD amplitudes?

# Why are Wilson loops with Lagrangian insertion in planar $\mathcal{N}=4$ sYM interesting?




[Alday, Buchbinder, Tseytlin '11]

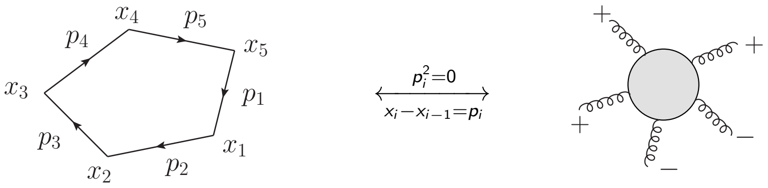
- Finite observable in planar $\mathcal{N}=4$ super Yang-Mills theory
- Very similar to massless scattering amplitudes in QCD
- Displays nice properities of $\mathcal{N}=4$ sYM amplitudes

  hidden symmetries and Yangian

  Grassmannian representation

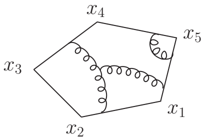
  positivity properties and Amplituhedron geometry

# Null Wilson loop / scattering amplitude duality in planar $\mathcal{N} = 4$ sYM




- Duality at the level of regularized quantities
- Duality at the level of their integrands

# Null Wilson loops




Light-light polygonal contour in 4D

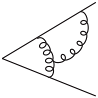
$$(x_i - x_{i+1})^2 = 0\,, \qquad i = 1, 2, \ldots, n$$

Null polygonal Wilson loop

$$W_n = \frac{1}{N_c} \mathrm{tr}_{\mathrm{F}} \mathrm{P}\exp\left[ i\, g_{\mathrm{YM}} \oint dx \cdot A(x) \right]$$

Cusp divergences of WL exponentiate (in dimensional regularisation $d = 4 - 2\epsilon$)

$$\log\langle W_n \rangle = \sum_{L \geq 1} \frac{g^{2L} \Gamma^{(L)}_{\mathrm{cusp}}}{(L\epsilon)^2} + \mathcal{O}\left(\frac{1}{\epsilon}\right)\,, \qquad g^2 \equiv \frac{N_c g^2_{\mathrm{YM}}}{16\pi^2}$$



Anomalous dual-conformal symmetry of $\langle W_n \rangle$ [Drummond, Henn, Korchemsky, Sokatchev '07]

$$x_i^\mu \to \frac{x_i^\mu}{x_i^2}$$

# Scattering amplitudes in a planar gauge theory

Massless scattering in a 4D gauge theory (in the adjoint representation)

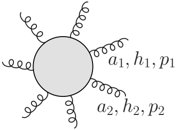
$$(a_i, h_i, p_i^\mu) \qquad \text{with} \qquad (p_i)^2 = 0\,, \qquad i = 1, 2, \dots, n$$

Maximal Helicity Violating (MHV) amplitudes

$$(h_1, h_2, \dots, h_n) = (+\dots+-+\dots+-+\dots+)$$

Planar limit $N_c \to \infty$ and $g_{\rm YM} \to 0$,

$$g^2 \equiv \frac{N_c g_{\rm YM}^2}{16\pi^2}$$




Color decomposition of the amplitudes

$$\mathcal{A}_n = g_{\rm YM}^{n-2} \sum_{\sigma \in S_n/Z_n} \text{tr}\left(T^{a_{\sigma_1}} \dots T^{a_{\sigma_n}}\right) A_n(g; \sigma(p_1^{h_1}), \dots, \sigma(p_n^{h_n})) + \mathcal{O}\left(N_c^{-1}\right)$$

# Planar MHV scattering amplitudes

Perturbative expansion of the planar amplitude

$$A_n^{\mathrm{MHV}} = A_{n,\mathrm{tree}}^{\mathrm{MHV}} + g^2 A_n^{(1)} + g^4 A_n^{(2)} + \ldots$$

Loop corrections $A_n^{(L)}$ require regularization of the soft-collinear divergences $\frac{1}{\epsilon^{2L}}$

Anomalous conformal symmetry of the amplitude

Exponentiation of the soft-collinear divergences and the ABDK/BDS ansatz

[Anastasiou, Bern, Dixon, Kosower '03][Bern, Dixon, Smirnov '05]

$$\log \frac{A_n^{\mathrm{MHV}}}{A_{n,\mathrm{tree}}^{\mathrm{MHV}}} \sim \underbrace{\log Z_{\mathrm{IR}}(g)}_{\epsilon\text{-poles}} + \frac{1}{4}\Gamma_{\mathrm{Cusp}}(g) \underbrace{F_n^{(1)}}_{\substack{\text{finite part}\\ \text{of 1-loop}\\ \text{amplitude}}} + \underbrace{R_n(g)}_{\substack{\text{finite}\\ \text{remainder}\\ \text{function}}}$$

The remainder function

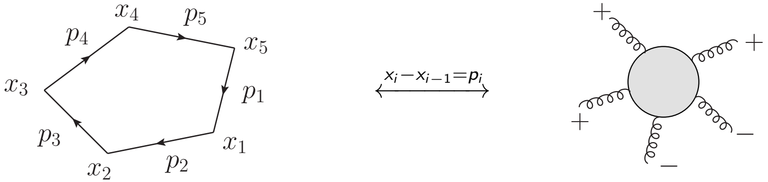
- vanishes in 4-particle and 5-particle scattering: $R_4 = R_5 = 0$
- Hexagon bootstrap program: $R_6$ is known up to seven loops, and $R_7$ is known up to four loops

[Caron-Huot, Dixon, Drummond, Dulat, Foster, Gurdogan, von Hippel, McLeod, Papathanasiou]

# Null Wilson loop / scattering amplitude duality of the regularized quantities

Cusp divergences of WL = IR divergences of MHV amplitude

Dual-conformal symmetry of WL is a hidden symmetry for MHV amplitude




[Alday, Maldacena '07] [Drummond, Korchemsky, Sokatchev '07][Brandhuber, Heslop, Travaglini '07]

$$\langle W_n \rangle \sim \frac{A_n^{\mathrm{MHV}}}{A_{n,\mathrm{tree}}^{\mathrm{MHV}}}$$

Divergences and anomalous symmetries, so better consider the duality at the level of the integrands

# Wilson loop integrands and Lagrangian insertion

How to define the integrand of the L-loop correction $W_n^{(L)}$ of the Wilson loop?

$$\langle W_n \rangle = 1 + g^2\, W_n^{(1)} + g^4\, W_n^{(2)} + \ldots\,, \qquad g^2 \equiv \frac{N_c g_{\rm YM}^2}{16\pi^2}$$

After rescaling of the fields $A^\mu \to \frac{1}{g_{\rm YM}} A^\mu$, ...

$$\langle W_n \rangle \sim \int \mathcal{D}A^\mu\, e^{\frac{i}{g_{\rm YM}^2}\int d^d y \mathcal{L}(y)}\, W_n$$

differentiation in the coupling constant results in the Lagrangian insertion formula

$$\boxed{g_{\rm YM}^2 \partial_{g_{\rm YM}^2} \langle W_n \rangle \sim \int d^d y\, \langle W_n \mathcal{L}(y) \rangle}$$

with *chiral on-shell* form of $\mathcal{N} = 4$ sYM Lagrangian, $\mathcal{L} = -\frac{1}{2}{\rm tr} F_{\alpha\beta} F^{\alpha\beta} + \ldots$

# Wilson loop integrands are Born-level correlators

$$\langle W_n \rangle = 1 + g^2\, W_n^{(1)} + g^4\, W_n^{(2)} + \ldots$$

Lagrangian insertion formula

$$g^2 \partial_{g^2} \langle W_n \rangle \sim \int d^d y\, \langle W_n \mathcal{L}(y) \rangle$$

One-loop correction

$$W_n^{(1)} \sim \int d^d y\, \underbrace{\langle W_n\, \mathcal{L}(y) \rangle_{\rm Born}}_{\text{one-loop integrand}}$$

L-loop correction

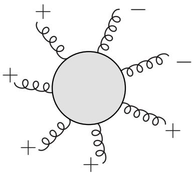
$$W_n^{(L)} \sim \int d^d y_1 \ldots d^d y_L\, \underbrace{\langle W_n\, \mathcal{L}(y_1) \ldots \mathcal{L}(y_L) \rangle_{\rm Born}}_{L\text{-loop integrand}}$$

The integrands are finite rational functions in four dimensions

# Planar amplitude integrands

$$M_n \equiv \frac{A_n^{\mathrm{MHV}}}{A_{n,\mathrm{tree}}^{\mathrm{MHV}}}$$

$$M_n = 1 + g^2\, M_n^{(1)} + g^4\, M_n^{(2)} + \ldots$$




Planar amplitudes in the dual-momenta variables

$$M_n^{(L)}(x_1,\ldots,x_n) \sim \int d^d y_1 \ldots d^d y_L\, \mathcal{I}_n^{(L)}(\underbrace{x_1,\ldots,x_n}_{\text{external dual momenta}} \mid \underbrace{y_1,\ldots,y_L}_{\text{loop dual momenta}})$$

The planar integrand is unique (assignment of loop variables $y_i$ is unambiguous)

# Four-dimensional amplitude integrands

Four-dimensional amplitude integrands $\mathcal{I}_n^{(L)}$ are generated via

- loop on-shell recurrence relations [Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka '10]
- Grassmannians and plabic graphs [Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka '12]
- Canonical form on positive geometry [Arkani-Hamed, Trnka '13]
- Twistor Feynman rules [Mason '05][Adamo, Boels, Bullimore, Mason, Skinner]
- Bootstrap from symmetries and analyticity [Bourjaily, DiRe, Shaikh, Spradlin, Volovich '11]

In principle, the integrands are known at any loop order

Four-particle integrands are explicitly known up to ten loop order [Bourjaily, Heslop, Tran '16]

# Null Wilson loop / scattering amplitude duality at the integrand level

$$\mathcal{I}^{(L)}_{\mathrm{MHV},n}(x_1,\ldots,x_n|y_1,\ldots,y_L) = \langle W[x_1,\ldots,x_n]\mathcal{L}(y_1)\ldots\mathcal{L}(y_L)\rangle_{\mathrm{Born}}$$

[Eden, Korchemsky, Sokatchev '10][Mason, Skinner '10]

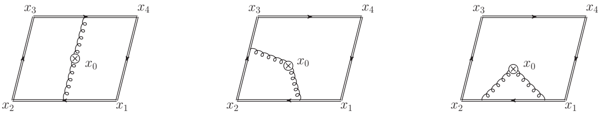
The integrands are

- planar
- four-dimensional
- rational functions with local poles
- dual-conformal invariant
- chiral: parity-even + parity-odd

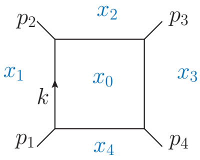
# Duality of the integrands: One-loop four-particle example

## Wilson loop integrand

$$\langle W[x_1, x_2, x_2, x_4]\mathcal{L}(x_0)\rangle_{\rm Born} = -\frac{x_{13}^2 x_{24}^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2}\,, \qquad x_{ij}^2 \equiv (x_i - x_j)^2$$




## MHV amplitude integrand




$$\mathcal{I}_4^{(1)}(x_1, x_2, x_3, x_4|x_0) = \frac{-(p_1+p_2)^2(p_2+p_3)^2}{k^2(k-p_1)^2(k+p_2)^2(k+p_2+p_3)^2}$$

$$= -\frac{x_{13}^2 x_{24}^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2}$$

# Null Wilson loop with Lagrangian insertion

$$F_n(x_1,\ldots,x_n;x_0) \equiv \frac{\langle W[x_1,\ldots,x_n]\mathcal{L}(x_0)\rangle}{\langle W[x_1,\ldots,x_n]\rangle}$$

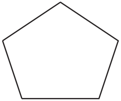
Finite observable. Cusp divergences cancel out in the ratio

Due to the Lagrangian insertion trick

$$g^2\partial_{g^2}\log\langle W_n\rangle \sim \int d^dx_0 \underbrace{F_n(x_1,\ldots,x_n;x_0)}_{\text{IR finite in 4D}} \sim \frac{1}{\epsilon^2}$$

L-loop $F_n$ is (L+1)-loop $\log\langle W_n\rangle$ with one of the loop integration frozen

# Null Wilson loop with Lagrangian insertion is an interesting observable with nice properties

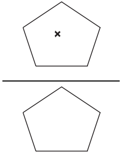
$\langle W_n \rangle =$ 

cusp divergences

transcendental function

of $3n - 15$ variables

anomalous
dual-conformal
symmetry

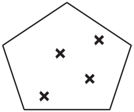
$F_n =$ 

finite in four dimensions

transcendental function

of $3n - 11$ variables

dual-conformal



$\langle W_n \, \mathcal{L}(y_1) \ldots \mathcal{L}(y_L) \rangle_{\text{Born}}$

finite in four dimensions

rational function

of $3n + 4L - 15$ variables

dual-conformal

# Structure of loop corrections

Perturbative weak-coupling expansion

$$F_n = \underbrace{g^2\, F_n^{(0)}}_{\text{Born-level}} + \underbrace{g^4\, F_n^{(1)}}_{\text{one-loop}} + \underbrace{g^6\, F_n^{(2)}}_{\text{two-loop}} + \ldots$$

L-loop correction

$$F_n^{(L)} = \sum_j \underbrace{R_{n,j}(x_1, \ldots, x_n; x_0)}_{\substack{\text{leading singularity} \\ \text{– rational function}}} \; \underbrace{g_{n,j}^{(L)}}_{\substack{\text{transcendental} \\ \text{function} \\ \text{weight } 2L}} \; \underbrace{(u_1, \ldots, u_{3n-11})}_{\substack{\text{dual-conformal} \\ \text{cross-ratios}}}$$

- What is the set of all-loop $n$-point leading singularities?
- What is the class of the transcendental functions for $n$-point observable?

# Structure of loop corrections

Cancel out dual-conformal weight $+4$ at the Lagrangian insertion point,

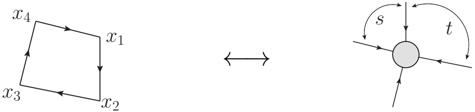
$$f_n(x_1,\ldots,x_n) \equiv \lim_{x_0\to\infty} (x_0^2)^4\, F_n(x_1,\ldots,x_n;x_0)$$

The frame choice breaks the dual-conformal symmetry

$$f_n(x_1,\ldots,x_n) \sim f_n(p_1,\ldots,p_n)$$

- Finite function in four dimensions
- Kinematics of $n$-particle scattering in a massless QFT
- Reminiscent of hard part of QCD amplitudes (highest transcendentality piece)

# Four-point observable




$$f_4^{(L)} = x_{13}^2 x_{24}^2 \, g^{(L)}\left(\frac{x_{13}^2}{x_{24}^2}\right) = \underbrace{s\,t}_{\text{leading sing.}} \; \underbrace{g^{(L)}}_{\text{HPL}}\left(\frac{t}{s}\right)$$

- One leading singularity $x_{13}^2 x_{24}^2$
- L-loop corrections are Harmonic Polylogarithms (HPL) of weight $2L$
- $f_4$ is known up to three loops

[Alday, Heslop, Sikorowski '12][Alday, Henn, Sikorowski '13][Henn, Korchemsky, Mistlberger '19]

$$g^{(0)}(z) = -1\,, \qquad g^{(1)}(z) = \log^2(z) + \pi^2\,, \qquad \ldots$$

at strong coupling [Alday, Buchbinder, Tseytlin '11], and the all-loop 'tree' part of 'loops of the loops' negative geometry expansion [Arkani-Hamed, Henn, Trnka '21]

# Five-point observable

$$f_5^{(L)} = f_5^{(0)}\, g_{5,0}^{(L)}(\mathrm{u}) + \sum_{i=1}^{5} r_{5,i}\, g_{5,i}^{(L)}(\mathrm{u})$$

- Six leading singularities [DC, Henn '22]

$$r_{5,0} \equiv f_5^{(0)}\,, \quad r_{5,1}\,, \quad r_{5,2}\,, \quad r_{5,3}\,, \quad r_{5,4}\,, \quad r_{5,5}$$

- Loop corrections $g_{5,i}^{(L)}$ are transcendental functions (pentagon functions) of four-variables

$$\mathrm{u} = \left\{ \frac{s_{12}}{s_{15}}, \frac{s_{23}}{s_{15}}, \frac{s_{34}}{s_{15}}, \frac{s_{45}}{s_{15}} \right\}, \qquad s_{ij} \equiv (p_i + p_j)^2$$

- Kinematics of a five-particle scattering amplitude in a massless QFT
- Nontrivial parity properties

$$i\epsilon_{\mu\nu\rho\sigma}\, p_1^\mu p_2^\nu p_3^\rho p_4^\sigma$$

- Explicit calculation up to two loops [DC, Henn '22]

# Five-particle leading singularities and conformal symmetry in momentum space

Spinor-helicity variables

$$p^2 = 0 \quad \Longrightarrow \quad p_\mu \sigma^\mu_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha}$$

and spinor products

$$\langle ij \rangle \equiv \lambda_i^\alpha \lambda_{j\alpha}\,, \qquad [ij] \equiv \tilde\lambda_{i\,\dot\alpha}\, \tilde\lambda_j^{\dot\alpha}$$

Six leading singularities normalized by the Parke-Taylor factor, $\mathrm{PT}_5\, r_{5,i}$, have a very simple form

$$\frac{[51]^2}{\langle 23\rangle\langle 34\rangle\langle 42\rangle} + \frac{[34]^2}{\langle 51\rangle\langle 12\rangle\langle 25\rangle} - \frac{[13]^2}{\langle 24\rangle\langle 45\rangle\langle 52\rangle}\,,\ \frac{[34]^2}{\langle 51\rangle\langle 12\rangle\langle 25\rangle}\,,\ \frac{[45]^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\,,\ \frac{[51]^2}{\langle 23\rangle\langle 34\rangle\langle 42\rangle}\,,\ \frac{[12]^2}{\langle 34\rangle\langle 45\rangle\langle 53\rangle}\,,\ \frac{[23]^2}{\langle 45\rangle\langle 51\rangle\langle 14\rangle}$$

Remarkable conformal symmetry (in momentum space)

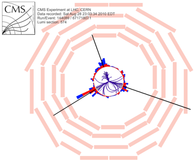
$$\mathbb{K}_{\alpha\dot\alpha} = \sum_{i=1}^{5} \frac{\partial^2}{\partial\lambda_i^\alpha \partial\tilde\lambda_i^{\dot\alpha}}\,, \qquad \mathbb{K}_{\alpha\dot\alpha}\Big( \underbrace{\mathrm{PT}_5}_{\frac{1}{\langle 12\rangle\langle 23\rangle\ldots\langle 51\rangle}}\; r_5 \Big) = 0$$

# Planar pentagon functions

$$dp_a^{(w)} = \sum_{k,b} A^k_{a,b}\, p_b^{(w-1)} d\log(W_k) + \text{terms with } p^{(w')} \text{ at } w' < w-1$$

26-letter planar pentagon alphabet

| | | |
|---|---|---|
| $W_1$ | $2\,p_1 \cdot p_2$ | $+(4)$ |
| $W_6$ | $2\,p_4 \cdot (p_3 + p_5)$ | $+(4)$ |
| $W_{11}$ | $2\,p_3 \cdot (p_4 + p_5)$ | $+(4)$ |
| $W_{16}$ | $2\,p_1 \cdot p_3$ | $+(4)$ |
| $W_{26}$ | $\frac{\mathrm{tr}[(1-\gamma_5)\not{p}_1\not{p}_2\not{p}_4\not{p}_5]}{\mathrm{tr}[(1+\gamma_5)\not{p}_1\not{p}_2\not{p}_4\not{p}_5]}$ | $+(4)$ |
| $W_{31}$ | $i\epsilon(p_1, p_2, p_3, p_4)$ | |




- pentagon functions $p_a^{(w)}$ of weight $w$
- classified at $w \leq 4$ [Gehrmann, Henn,Lo Presti '18]

| $w$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| # | 1 | 5 | 5 | 16 | 56 |

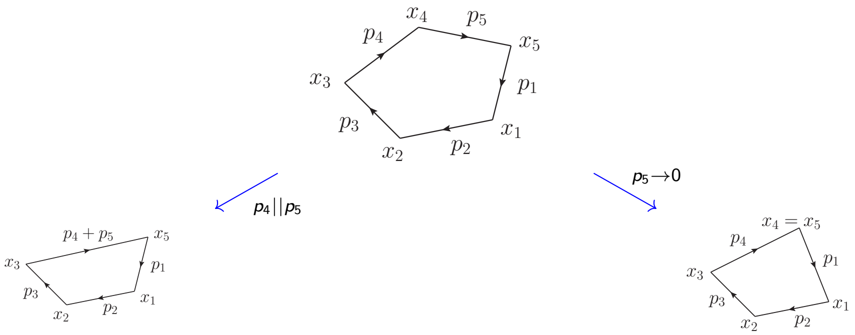
- algebraically independent
- polylogarithmic iterated integral representation

$$\int d\log(W_{i_1}) \dots d\log(W_{i_w})$$

- routines for numerical evaluations
- calculation of QCD corrections for $2 \to 3$ production at NNLO (leading color) [Chawdhry, Czakon, Mitov, Poncelet '19, '21][Kallweit, Sotnikov, Wiesemann '20] and nonplanar extension of the pentagon functions [DC, Sotnikov '20]

# Soft and collinear limits of the five-point observable are smooth

$$f_5 \to f_4 \quad \text{at} \quad p_5 \to 0 \quad \text{or} \quad p_4 || p_5$$



# Definite sign of the loop corrections

Amplituhedron program: integrands of amplitudes are volume forms, positivity of amplitude integrands [Arkani-Hamed, Trnka '13]

Positivity of the integrated loop corrections? Observed for $f_4^{(L)}$ [Arkani-Hamed, Henn, Trnka '21]

We observe five-particle positivity (at $L = 0, 1, 2$) [DC, Henn '22]

$$(-1)^{L+1} f_5^{(L)} > 0$$

in the one-loop Amplituhedron region of the five-particle scattering

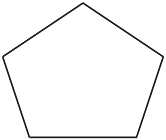
$$s_{12} < 0\,,\ s_{23} < 0\,, \ldots, s_{15} < 0\,, \quad i\epsilon_{\mu\nu\rho\sigma} p_1^{\mu} p_2^{\nu} p_3^{\rho} p_4^{\sigma} > 0$$

Highly nontrivial, since the sign of individual terms

$$f_5^{(L)} = r_5^{(0)}\, g_{5,0}^{(L)}(\mathrm{u}) + \ldots + r_{5,5}\, g_{5,5}^{(L)}(\mathrm{u})$$

varies inside the Amplituhedron region.

# Duality with all-plus amplitude in pure Yang-Mills



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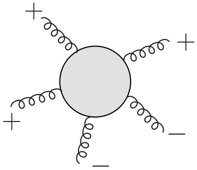


$\mathcal{N} = 4$ super-Yang-Mills

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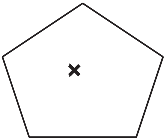
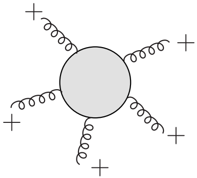


$\overset{?}{\leftrightarrow}$

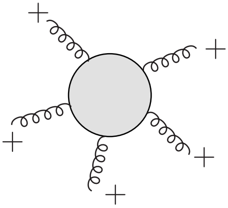



$\mathcal{N} = 4$ super-Yang-Mills

pure Yang-Mills

# Planar all-plus amplitude in pure Yang-Mills

$$\mathcal{A}_n^{\rm YM} = g_{\rm YM}^{n-2} \sum_{L\geq 1} g^{2L} \sum_{\sigma\in S_n/Z_n} \mathrm{tr}\left(T^{a_{\sigma_1}} \ldots T^{a_{\sigma_n}}\right) A_{{\rm YM},n}^{(L)}(\sigma_1^+, \ldots, \sigma_n^+) + \mathcal{O}\left(N_c^{-1}\right)$$




tree-level $A_{{\rm YM},n}^{(0)} = 0$

one-loop $A_{{\rm YM},n}^{(1)}$ is a finite rational function

two-loop $A_{{\rm YM},n}^{(2)}$ is similar to a one-loop QCD amplitude (highest transcendental weight two)

three-loop $A_{{\rm YM},n}^{(3)}$ is similar to a two-loop QCD amplitude (highest transcendental weight four)

# Planar all-plus amplitude in pure Yang-Mills

$$A_n^{\rm YM} = \underbrace{\mathcal{Z}_{\rm IR}^{\rm YM}}_{\text{IR poles}} \underbrace{g^2 A_{\rm YM,n}^{(1)}}_{\text{Born level}} \underbrace{\mathcal{H}_n^{\rm YM}}_{\text{finite part}} + \mathcal{O}\left(N_c^{-1}\right)$$

Finite hard part of the all-plus amplitude

$$\mathcal{H}_n^{\rm YM} = 1 + g^2 \mathcal{H}_{{\rm YM},n}^{(1)} + g^4 \mathcal{H}_{{\rm YM},n}^{(2)} + \mathcal{O}(g^6)$$

Available perturbative data (planar and nonplanar)

- one-loop $n$-particle [Bern, Chalmers, Dixon, Kosower '93][Henn, Power, Zoia '19]
- two-loop five-particle [Gehrmann, Henn, Lo Presti '15] [Badger, DC, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, Zoia '18]
- two-loop $n$-particle [Dunbar, Jehu, Perkins '16][Dunbar, Godwin, Jehu, Perkins '17] [Dunbar, Perkins, Strong '20]
- three-loop four-particle [Jin, Luo '19][Caola, Chakraborty, Gambuti, von Manteuffel, Tancredi '21]

# Duality planar all-plus amplitude / Wilson loop with Lagrangian insertion

Duality at the lowest perturbative order

$$\mathrm{PT}_n\, f_n^{(0)} = A^{(1)}_{\mathrm{YM},n}$$

Duality at the loop level (in 4D and at the highest transcendentality level)

$$\log \mathcal{H}_n^{\mathrm{MHV}} + \log\left(\frac{f_n}{g^2 f_n^{(0)}}\right) \sim \log \mathcal{H}_n^{\mathrm{YM}} + \mathcal{O}(\epsilon)$$

where $\mathcal{H}_n^{\mathrm{MHV}}$ is the finite part of MHV amplitude (BDS ansatz at $n \leq 5$)

Agreement with available perturbative data for the all-plus YM amplitude

Predictions for the all-plus YM amplitude (planar and maximal transcendentality)

- three-loop five-particle
- four-loop four-particle

# n-particle leading singularities

$$f_n^{(L)} = \sum_j r_{n,j}\, g_{n,j}^{(L)}(\mathrm{u})$$

Conjecture: there are $(n-1)(n-2)^2(n-3)/12$ leading singularities $r_{n,j}$ of the $n$-particle $f_n$. They are residues of the contour integral over Grassmannian $\mathrm{Gr}(2,n)$

[DC, Henn '22]

$$r_n = \int \frac{d^{2n}C}{\mathrm{vol}(GL(2))} \frac{\left(\sum_{i<j} (\vec{c}_i \vec{c}_j)\langle i|x_i x_j|j\rangle\right)^2}{(\vec{c}_1\vec{c}_2)(\vec{c}_2\vec{c}_3)\ldots(\vec{c}_n\vec{c}_1)} \delta^{2\times 2}\left(\sum_{i=1}^{n} \vec{c}_i |i\rangle\right)$$

Confirm by explicit calculation at the Born level and one-loop: $f_n^{(0)}$ and $f_n^{(1)}$

Conformal symmetry (in momentum space)

$$\mathbb{K}_{\alpha\dot\alpha} = \sum_{i=1}^{n} \frac{\partial^2}{\partial\lambda_i^\alpha \partial\tilde\lambda_i^{\dot\alpha}}\,, \qquad \mathbb{K}_{\alpha\dot\alpha}\Big( \underbrace{\mathrm{PT}_n}_{\frac{1}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}}\; r_n \Big) = 0$$

# What are the symmetries of the leading singularities?

$$\begin{array}{ccccc} \text{Conformal} & & \text{Dual-conformal} & & \text{Yangian} \\ \mathbb{K}_\mu,\ \mathbb{P}_\mu,\ \mathbb{M}_{\mu\nu},\ \mathbb{D} & + & \mathrm{K}_\mu,\ \mathrm{P}_\mu,\ \mathrm{M}_{\mu\nu},\ \mathrm{D} & = & \mathrm{Y}(su(2,2)) \end{array}$$

but $K_\mu$ is broken by the choice $x_0 = \infty$

- Poincaré part of the dual conformal symmetry: $M_{\mu\nu},\ P_\mu,\ D$
- Conformal symmetry (hidden symmetry for $f_n$)
- Other Yangian symmetries?

# Dual conformal generators in momentum twistor variables

Momentum twistors

$$Z^I = \begin{pmatrix} \lambda^\alpha \\ \mu^{\dot\alpha} \end{pmatrix} , \qquad \mu^{\dot\alpha} \equiv x^{\dot\alpha\beta}\lambda_\beta$$

Fifteen dual-conformal generators

$$\mathfrak{J}^I{}_J = Z^I \frac{\partial}{\partial Z^J}$$

at zero central charge

$$\lambda^\alpha \frac{\partial}{\partial \lambda^\alpha} + \mu^{\dot\alpha} \frac{\partial}{\partial \mu^{\dot\alpha}} = 0$$

form a $4 \times 4$ matrix

$$\mathfrak{J}^I{}_J = \begin{pmatrix} M^\alpha{}_\beta - \frac{\delta^\alpha_\beta}{2} D & P^\alpha{}_{\dot\beta} \\ K^{\dot\alpha}{}_\beta & \bar{M}^{\dot\alpha}{}_{\dot\beta} + \frac{\delta^{\dot\alpha}_{\dot\beta}}{2} D \end{pmatrix}$$

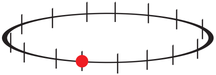
$I = \alpha, \dot\alpha$ and $J = \beta, \dot\beta$

# Yangian generators in dual momenta variables

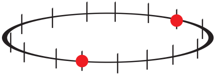
local dual-conformal generators

$$[\mathfrak{J}_i]^I{}_J = Z_i^I \frac{\partial}{\partial Z_i^J}\,, \qquad i = 1, \ldots, n$$

level-0 $\quad \mathfrak{J}^I{}_J \equiv \left[\mathfrak{J}^{(0)}\right]^I{}_J = \sum_{i=1}^{n} [\mathfrak{J}_i]^I{}_J$



level-1 $\quad \widehat{\mathfrak{J}}^I{}_J \equiv \left[\mathfrak{J}^{(1)}\right]^I{}_J = \sum_{1\le i<j\le n} [\mathfrak{J}_i]^I{}_K\, [\mathfrak{J}_j]^K{}_J$



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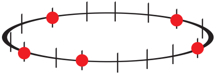
level-k $\quad \left[\mathfrak{J}^{(k)}\right]^I{}_J = \sum_{1\le i_1<i_2<\ldots<i_{k+1}\le n} \left(\mathfrak{J}_{i_1}\right)^I{}_{K_1} \left(\mathfrak{J}_{i_2}\right)^{K_1}{}_{K_2} \ldots \left(\mathfrak{J}_{i_{k+1}}\right)^{K_k}{}_J$



# Yangian symmetries of the leading singularities

Level-zero

Yes : $\mathrm{P}_{\mu}\,,\,\mathrm{M}_{\mu\nu}\,,\,\mathrm{D}\;\sim\;\mathfrak{J}^{\alpha}{}_{\beta}\,,\,\mathfrak{J}^{\alpha}{}_{\dot\beta}\,,\,\mathfrak{J}^{\dot\alpha}{}_{\dot\beta}$ No : $\mathrm{K}_{\mu}\sim\mathfrak{J}^{\dot\alpha}{}_{\beta}$

Level-one

Yes : $\widehat{\mathfrak{J}}^{\alpha}{}_{\beta}\,,\,\widehat{\mathfrak{J}}^{\alpha}{}_{\dot\beta}\sim\mathbb{K}^{\alpha}{}_{\dot\beta}\,,\,\widehat{\mathfrak{J}}^{\dot\alpha}{}_{\dot\alpha}$ No : $\widehat{\mathfrak{J}}^{(\dot\alpha\dot\beta)}\,,\,\widehat{\mathfrak{J}}^{\dot\alpha}{}_{\beta}$

Level-$k$ at $k\geq 2$

Yes : $\left(\mathfrak{J}^{(k)}\right)^{\alpha}{}_{\beta}\,,\;\left(\mathfrak{J}^{(k)}\right)^{\dot\alpha}{}_{\beta}\,,\;\left(\mathfrak{J}^{(k)}\right)^{\alpha}{}_{\dot\beta}\,,\;\left(\mathfrak{J}^{(k)}\right)^{\dot\alpha}{}_{\dot\beta}$

The Yangian symmetries also take place for the leading singularities at arbitrary $x_0$

$$R_{n,j}(x_1,\ldots,x_n;x_0)$$

# Conclusions

Wilson loop with Lagrangian insertion has similarities with finite parts of massless scattering amplitudes in QCD

Several remarkable properties:

- Positivity in the Amplituhedron region
- Duality with all-plus amplitudes in pure Yang-Mills
- Conformal symmetry of the leading singularities

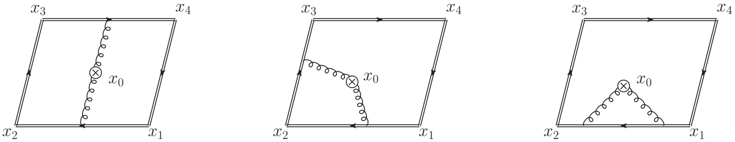
# Backup slides

# Perturbative expansion

$$F_n = \underbrace{g^2\, F_n^{(0)}}_{\text{Born-level}} + \underbrace{g^4\, F_n^{(1)}}_{\text{one-loop}} + \underbrace{g^6\, F_n^{(2)}}_{\text{two-loop}} + \dots$$

Born-level

$F_4^{(0)}$ :




Finite dual-conformal rational functions, e.g.

$$F_4^{(0)} = -\frac{x_{13}^2 x_{24}^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2}\,,$$

$$F_5^{(0)} = -\frac{1}{2x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2 x_{50}^2}\Big[x_{24}^2 x_{35}^2 x_{10}^2 + x_{14}^2 x_{35}^2 x_{20}^2 + x_{14}^2 x_{25}^2 x_{30}^2 + x_{13}^2 x_{25}^2 x_{40}^2 + x_{13}^2 x_{24}^2 x_{50}^2 + \epsilon_{123450}\Big]$$

Nontrivial parity properties at $n \geq 5$

# The Born-level approximation and n-particle leading singularities

four-particle

$$f_4^{(0)} = b_{1234} = -x_{13}^2 x_{24}^2$$

five-particle

$$\begin{aligned} f_5^{(0)} &= b_{1245} + b_{2345} - b_{12345} \\ &= -\frac{1}{2}\left[x_{24}^2 x_{35}^2 + x_{14}^2 x_{35}^2 + x_{14}^2 x_{25}^2 + x_{13}^2 x_{25}^2 + x_{13}^2 x_{24}^2 + 4i\epsilon(x_{12}, x_{23}, x_{34}, x_{45})\right] \end{aligned}$$

$n$-particle

$$f_n^{(0)} = \sum_{i=1}^{n-3} b_{i,i+1,n-1,n} - \sum_{i=1}^{n-4} b_{i,i+1,n-2,n-1,n} - \sum_{k=2}^{n-4} \sum_{i=1}^{n-k-3} b_{i,i+1,i+k,i+k+1,n}$$

Spurious poles of $b$'s cancel out among each other

# n-particle leading singularities

Leading singularities of $f_n$ are linear combinations of 4-point and 5-point functions

$$b_{ijkl} = \frac{\big(\langle ijkl\rangle\big)^2}{\langle ij\rangle\langle jk\rangle\langle kl\rangle\langle li\rangle}\,, \qquad 1 \le i < j < k < l \le n$$

$$b_{ijklm} = \frac{\big(\langle ij\rangle\langle jklm\rangle - \langle jm\rangle\langle ijkl\rangle\big)^2}{\langle ij\rangle\langle jk\rangle\langle kl\rangle\langle kj\rangle\langle jm\rangle\langle mi\rangle}\,, \qquad 1 \le i < j < k < l < m \le n$$

of the momentum twistor variables (infinity bi-twistor is fixed!)

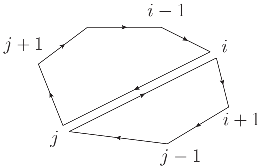
$$x_i - x_{i-1} = p_i = |i\rangle[i|\,, \qquad Z_i = \begin{pmatrix} |i\rangle \\ x_i|i\rangle \end{pmatrix}$$

with the twistor 4-bracket

$$\langle abcd\rangle \equiv \langle a|x_a x_b|b\rangle\langle cd\rangle - \langle a|x_a x_c|c\rangle\langle b\rangle + \langle a|x_a x_d|d\rangle\langle bc\rangle + \langle b|x_b x_c|c\rangle\langle ad\rangle - \langle b|x_b x_d|d\rangle\langle ac\rangle + \langle c|x_c x_d|d\rangle\langle ab\rangle$$

- "Kermits" (1-loop integrand of MHV) [Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka '10]
- zero helicity weight ⇔ $b$'s are functions of coordinates $x_1, \ldots, x_n$
- dual-Poincaré invariant
- contain spurious poles

# One-loop n-particle observable

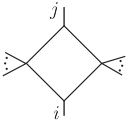



$$f_n^{(0)}(1,2,\ldots,n) = \sum_{i=1}^{n-3} b_{i,i+1,n-1,n} - \sum_{i=1}^{n-4} b_{i,i+1,n-2,n-1,n} - \sum_{k=2}^{n-4}\sum_{i=1}^{n-k-3} b_{i,i+1,i+k,i+k+1,n}$$

The local integrand form of the two-loop $\log A_n^{\mathrm{MHV}}$ [Arkani-Hamed, Bourjaily, Cachazo, Trnka '10] integrates into

$$f_n^{(1)} = \sum_{\substack{1\leq i<j\leq n \\ 1<|i-j|<n-1}} \Big[ f_n^{(0)}(1,2,\ldots,n) - f_{j-i+1}^{(0)}(i,i+1,\ldots,j-1,j) - f_{n+1+i-j}^{(0)}(j,j+1,\ldots,i-1,i) \Big] F_{ij}^{\mathrm{2me}}\left(\frac{x_{i-1\,j}^2}{x_{ij}^2}, \frac{x_{i\,j-1}^2}{x_{ij}^2}, \frac{x_{i-1\,j-1}^2}{x_{ij}^2}\right)$$

Two-mass easy-box (four-dimensional finite pure function: $\mathrm{Li}_2$ and $\log \times \log$)

$F_{ij}^{\mathrm{2me}} =$ 

# Five-particle leading singularities

Six leading singularities $r_5$:

$$b_{1234}\,,\quad b_{1245}\,,\quad b_{1235}\,,\quad b_{1345}\,,\quad b_{2345}\,,\quad b_{12345}$$

Remarkable conformal symmetry (in momentum space)

$$\mathrm{PT}_5\, b_{1234} = -\frac{[23]^2}{\langle 45\rangle\langle 51\rangle\langle 14\rangle}\,, \qquad \mathrm{PT}_5\, b_{2345} = -\frac{[34]^2}{\langle 51\rangle\langle 12\rangle\langle 25\rangle}\,,$$

$$\mathrm{PT}_5\, b_{1345} = -\frac{[45]^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\,, \qquad \mathrm{PT}_5\, b_{1245} = -\frac{[51]^2}{\langle 23\rangle\langle 34\rangle\langle 42\rangle}\,,$$

$$\mathrm{PT}_5\, b_{1235} = -\frac{[12]^2}{\langle 34\rangle\langle 45\rangle\langle 53\rangle}\,, \qquad \mathrm{PT}_5\, b_{12345} = \frac{[13]^2}{\langle 24\rangle\langle 45\rangle\langle 52\rangle}$$

Conformal boost annihilates the leading singularities (normalised by the Parke-Taylor factor $\mathrm{PT}_5$)

$$\underbrace{\mathbb{K}_{\alpha\dot\alpha}}_{\sum_{i=1}^{5} \frac{\partial^2}{\partial\lambda_i^{\alpha}\partial\tilde\lambda_i^{\dot\alpha}}} \quad (\mathrm{PT}_5\, r_5) = 0$$

# Yangian symmetries of the leading singularities

Level-zero (Poincaré part of the dual conformal symmetry)

$$\mathfrak{J}^{\alpha}{}_{\beta}\, r_n = -2\delta^{\alpha}_{\beta}\, r_n\,, \qquad \mathfrak{J}^{\alpha}{}_{\dot\beta}\, r_n = 0\,, \qquad \mathfrak{J}^{\dot\alpha}{}_{\dot\beta}\, r_n = 2\delta^{\dot\alpha}_{\dot\beta} r_n$$

Level-one

$$\widehat{\mathfrak{J}}^{\alpha}{}_{\beta}\, r_n = \delta^{\alpha}_{\beta}\, r_n\,, \qquad \widehat{\mathfrak{J}}^{\alpha}{}_{\dot\beta}\, r_n = 0\,, \qquad \widehat{\mathfrak{J}}^{\dot\alpha}{}_{\dot\alpha}\, r_n = -2 r_n$$

Level-$k$,

$$\left(\mathfrak{J}^{(k)}\right)^{\alpha}{}_{\beta}\, r_n = \left(\mathfrak{J}^{(k)}\right)^{\dot\alpha}{}_{\beta}\, r_n = \left(\mathfrak{J}^{(k)}\right)^{\alpha}{}_{\dot\beta}\, r_n = \left(\mathfrak{J}^{(k)}\right)^{\dot\alpha}{}_{\dot\beta}\, r_n = 0$$

with $k = 2, \ldots, n-1$

Broken Yangian symmetries

$$\mathfrak{J}^{\dot\alpha}{}_{\beta}\,, \quad \widehat{\mathfrak{J}}^{(\dot\alpha\dot\beta)}\,, \quad \widehat{\mathfrak{J}}^{\dot\alpha}{}_{\beta}$$

Conformal boost generator among Yangian level-one generators

$$0 = \widehat{\mathfrak{J}}^{\alpha}{}_{\dot\beta}\, r_n(Z_1, \ldots, Z_n) = -\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle \underbrace{\sum_{i=1}^{n} \frac{\partial^2}{\partial\lambda_i^{\alpha}\partial\tilde\lambda_i^{\dot\alpha}}}_{\mathbb{K}_{\alpha\dot\alpha}} \left[\frac{r_n(p_1, \ldots, p_n)}{\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle}\right]$$

# Spin chain picture of the Yangian symmetry

BCFW-like decomposition of the leading singularities

$$r_n(Z_1,\ldots,Z_n) = \underbrace{\mathrm{R}_{32}\,\mathrm{R}_{43}\,\ldots\mathrm{R}_{n\,n-1}\,\mathrm{R}_{n-2\,n-1}\,\ldots\mathrm{R}_{23}\,\mathrm{R}_{12}}_{2(n-2)\text{ "excitations"}}\;\underbrace{|\Omega_n\rangle}_{\text{"vacuum"}}$$

"Vacuum" state

$$|\Omega_n\rangle := \delta^2(\lambda_1)\delta^2(\lambda_n)[\mu_1\mu_n]^2$$

BCFW-bridge

$$[\mathrm{R}_{ij}\,G]\,(Z_i,Z_j) := \int \frac{dt}{t}\,G(Z_i+tZ_j,Z_j)$$

Lax operator is a $4\times 4$ matrix of dual-conformal generators (spectral parameter $u$)

$$[\mathrm{L}(u)]^I{}_J := u\,\delta^I_J + Z^I\frac{\partial}{\partial Z^J}$$

Monodromy matrix operator (matrix of the Yangian generators)

$$\begin{aligned}[\mathrm{T}(u)]^I{}_J &:= [\mathrm{L}_1(u)]^I{}_{K_1}\,[\mathrm{L}_2(u)]^{K_1}{}_{K_2}\ldots[\mathrm{L}_n(u)]^{K_{n-1}}{}_J\\ &= \sum_{k=-1}^{n-1} u^{n-k-1}\left[\mathfrak{J}^{(k)}\right]^I{}_J\end{aligned}$$

# Spin chain picture of the Yangian symmetry

Local commutation relations

$$[\mathrm{R}_{ij}\,,\,\mathrm{L}_i(u)\,\mathrm{L}_j(u)] = [\mathrm{R}_{ji}\,,\,\mathrm{L}_i(u)\,\mathrm{L}_j(u)] = 0$$

Global commutation relations

$$[\mathrm{R}_{i\,i+1}\,,\,\mathrm{T}(u)] = [\mathrm{R}_{i+1\,i}\,,\,\mathrm{T}(u)] = 0$$

Yangian-like symmetries of the leading singularities

$$\mathrm{T}(u)\,\underbrace{\mathrm{R}_{32}\ldots\mathrm{R}_{12}\,|\Omega_n\rangle}_{r_n} = \mathrm{R}_{32}\ldots\mathrm{R}_{12}\,\mathrm{T}(u)\,|\Omega_n\rangle$$

$$= u^{n-2}\begin{pmatrix} u^2-2u+1 & 0 \\ 0 & u^2+2u-1 \end{pmatrix} r_n + u^{n-2}\begin{pmatrix} 0 & 0 \\ \mathcal{O}(u) & \mathcal{O}(u^0) \end{pmatrix}$$